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Families of Transformations of Straight Lines into Spheres.

BY E. O. LOVETT.

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1. By the transformations of the point-space (x, y, z) into the point-space (X, Y, Z), which are determined by two *æquationes directrices*

$$\Phi(x, y, z, X, Y, Z) = 0, \quad \Psi(x, y, z, X, Y, Z) = 0, \tag{1}$$

the straight line

$$y + kx + m = 0, \quad z + lx + n = 0, \tag{2}$$

is changed into the surface

$$\Omega(X, Y, Z, k, l, m, n) = 0, \tag{3}$$

whose equation is obtained by eliminating x, y, z by means of the four equations (1) and (2).

If the equations (1) are of the form

$$x\Phi_1 + y\Phi_2 + z\Phi_3 + \Phi_4 = 0, \quad x\Phi_5 + y\Phi_6 + z\Phi_7 + \Phi_8 = 0, \tag{4}$$

where the Φ_i's are any functions of X, Y, Z not containing x, y, z, the line (2) will be transformed into the surface

$$\begin{vmatrix} \Phi_1 & \Phi_2 & \Phi_3 & \Phi_4 \\ \Phi_5 & \Phi_6 & \Phi_7 & \Phi_8 \\ k & 1 & 0 & m \\ l & 0 & 1 & n \end{vmatrix} = 0. \tag{5}$$

In order that this surface be a quadric for all values of k, l, m, n, it is necessary and sufficient that all the determinants of the matrix:

$$\left\| \begin{matrix} \Phi_1 & \Phi_2 & \Phi_3 & \Phi_4 \\ \Phi_5 & \Phi_6 & \Phi_7 & \Phi_8 \end{matrix} \right\| \tag{6}$$

reduce to functions of degree not higher than the second.

2. In particular, let all of the functions Φ_i be linear, then the equations

$$x\phi_1 + y\phi_2 + z\phi_3 + \phi_4 = 0, \quad x\phi_5 + y\phi_6 + z\phi_7 + \phi_8 = 0, \tag{7}$$

where
$$\phi_i \equiv a_i X + b_i Y + c_i Z + d_i \quad (a_i,\ b_i,\ c_i,\ d_i \text{ constants}) \tag{8}$$

define a family of ∞^{30} transformations which change the straight line (2) into the quadric

$$\begin{vmatrix} k & m \\ l & n \end{vmatrix} \cdot \begin{vmatrix} \phi_2 & \phi_3 \\ \phi_6 & \phi_7 \end{vmatrix} - k \begin{vmatrix} \phi_2 & \phi_4 \\ \phi_6 & \phi_8 \end{vmatrix} - l \begin{vmatrix} \phi_3 & \phi_4 \\ \phi_7 & \phi_8 \end{vmatrix} - m \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_5 & \phi_6 \end{vmatrix} - n \begin{vmatrix} \phi_1 & \phi_3 \\ \phi_5 & \phi_7 \end{vmatrix} + \begin{vmatrix} \phi_1 & \phi_4 \\ \phi_5 & \phi_8 \end{vmatrix} = 0. \tag{9}$$

If we demand that this quadric shall be a sphere, certain well-known relations must exist among the coefficients of the terms of the second degree; by means of these relations we can express the conditions to be satisfied by the defining constants of the linear functions (8) in order that the transformations determined by the two bilinear equations (7) may be line-sphere transformations. It is clear that if every transformation of the family defined by the equations (7) is to transform every straight line into a sphere, then the above-named equations of condition must not contain the parameters k, l, m, n.

The form of the equation (9) shows that the quadric reduces to a sphere without k, l, m, n entering the equations of condition in the following cases and in no other:

1° when any determinant of the matrix

$$\left\| \begin{matrix} \psi_1 & \psi_2 & \psi_3 & \psi_4 \\ \psi_5 & \psi_6 & \psi_7 & \psi_8 \end{matrix} \right\|, \tag{10}$$

where
$$\psi_i \equiv a_i X + b_i Y + c_i Z \equiv \phi_i - d_i \tag{11}$$
reduces to the form
$$\text{const.}\ (X^2 + Y^2 + Z^2), \tag{12}$$

and at the same time the functions ϕ_i corresponding to the ψ_i remaining in the matrix reduce to constants; for example, $\phi_2, \phi_3, \phi_6, \phi_7$ constants and the determinant $\psi_1\psi_8 - \psi_4\psi_5$ of the form (12);

2° when all six determinants of the matrix (10) are of the form (12).

3. Examining the first case, and with particular reference to the example noted, we find that the two bilinear equations

$$\left.\begin{aligned}(a_1 X + b_1 Y + c_1 Z + d_1)\,x + d_2 y + d_3 z + a_4 X + b_4 Y + c_4 Z + d_4 = 0,\\ (a_5 X + b_5 Y + c_5 Z + d_5)\,x + d_6 y + d_7 z + a_8 X + b_8 Y + c_8 Z + d_8 = 0,\end{aligned}\right\} \tag{13}$$

where the constants are subject to the conditions

$$\left.\begin{aligned}&a_1 a_8 - a_4 a_5 = b_1 b_8 - b_4 b_5 = c_1 c_8 - c_4 c_5,\\ &a_1 b_8 + b_1 a_8 - a_4 b_5 - b_4 a_5 = 0,\\ &b_1 c_8 + c_1 b_8 - b_4 c_5 - c_4 b_5 = 0,\\ &c_1 a_8 + a_1 c_8 - c_4 a_5 - a_4 c_5 = 0,\end{aligned}\right\} \tag{14}$$

determine a family of ∞^{13} transformations which change all straight lines into spheres. The exponent is readily verified since each equation of (13) is linear and homogeneous in ten constants which are connected by five independent equations (14).

Treating the remaining determinants of the matrix and the corresponding functions in the same manner, we obtain six families of ∞^{13} transformations changing straight lines into spheres.

4. By employing the method of Sophus Lie,* we can show that the preceding line-sphere transformations are contact transformations.

The finite equations expressing the coordinates X, Y, Z, P, Q of the surface element corresponding to the surface element (x, y, z, p, q) as functions of the coordinates of the latter, are given by the resolution of a system of linear equations composed of the two following:

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* Lie-Engel, "Theorie der Transformationsgruppen," vol. 2.

$$\left.\begin{aligned} \Omega_1 \equiv (a_1 x + a_4) X + (b_1 x + b_4) Y + (c_1 x + c_4) Z & \\ + d_1 x + d_2 y + d_3 z + d_4 = 0, & \\ \Omega_2 \equiv (a_5 x + a_8) X + (b_5 x + b_8) Y + (c_5 x + c_8) Z & \\ + d_5 x + d_6 y + d_7 z + d_8 = 0, & \end{aligned}\right\} \quad (15)$$

and the three obtained by eliminating the ratio $\lambda_1 : \lambda_2$ from the system

$$\left.\begin{aligned} \left(\lambda_1 \frac{\partial \Omega_1}{\partial z} + \lambda_2 \frac{\partial \Omega_2}{\partial z}\right) p + \lambda_1 \frac{\partial \Omega_1}{\partial x} + \lambda_2 \frac{\partial \Omega_2}{\partial x} &= 0, \\ \left(\lambda_1 \frac{\partial \Omega_1}{\partial z} + \lambda_2 \frac{\partial \Omega_2}{\partial z}\right) q + \lambda_1 \frac{\partial \Omega_1}{\partial y} + \lambda_2 \frac{\partial \Omega_2}{\partial y} &= 0, \\ \left(\lambda_1 \frac{\partial \Omega_1}{\partial Z} + \lambda_2 \frac{\partial \Omega_2}{\partial Z}\right) P + \lambda_1 \frac{\partial \Omega_1}{\partial X} + \lambda_2 \frac{\partial \Omega_2}{\partial X} &= 0, \\ \left(\lambda_1 \frac{\partial \Omega_1}{\partial Z} + \lambda_2 \frac{\partial \Omega_2}{\partial Z}\right) Q + \lambda_1 \frac{\partial \Omega_1}{\partial Y} + \lambda_2 \frac{\partial \Omega_2}{\partial Y} &= 0; \end{aligned}\right\} \quad (16)$$

in fact X, Y, Z are found by solving the system

$$\Omega_1 = 0, \quad \Omega_2 = 0, \quad AX + BY + CZ + D = 0, \quad (17)$$

where

$$\left.\begin{aligned} & A = a_5 e - a_1 f, \quad B = b_5 e - b_1 f, \quad C = c_5 e - c_1 f, \\ & D = d_5 e - d_1 f - (d_3 d_6 - d_2 d_7) p, \quad e = d_2 + d_3 q, \quad f = d_6 + d_7 q; \end{aligned}\right\} \quad (18)$$

and P, Q have the values

$$\left.\begin{aligned} P &= \frac{1}{R} \{(a_1 x + a_4)(d_6 + d_7 q) - (a_5 x + a_8)(d_2 + d_3 q)\}, \\ Q &= \frac{1}{R} \{(b_1 x + b_4)(d_6 + d_7 q) - (b_5 x + b_8)(d_2 + d_3 q)\}, \\ R &= (c_5 x + c_8)(d_2 + d_3 q) - (c_1 x + c_4)(d_6 + d_7 q). \end{aligned}\right\} \quad (19)$$

5. Relative to the transformations defined by the equations (13) and (14), it

is interesting to observe further that the absolute term of the equation (9) becomes zero independently of k, l, m, n if the equations

$$\left.\begin{array}{lll} d_1 d_6 - d_2 d_5 = 0, & d_1 d_7 - d_3 d_5 = 0, & d_1 d_8 - d_4 d_5 = 0, \\ d_2 d_7 - d_3 d_6 = 0, & d_2 d_8 - d_4 d_6 = 0, & d_3 d_8 - d_4 d_7 = 0, \end{array}\right\} \tag{20}$$

hold; then the transformations determined by (13) and (14) change straight lines into points if the additional conditions

$$d_1 : d_2 : d_3 : d_4 = d_5 : d_6 : d_7 : d_8 \tag{21}$$

are satisfied.

The following particular case is especially noteworthy. Assigning the values

$$\left.\begin{array}{l} a_1 = a_8 = b_1 = b_8 = c_4 = c_5 = d_1 = d_2 = d_4 = d = d_7 = d_8 = 0, \\ a_4 = a_5 = c_1 = -c_8 = d_3 = d_6 = 1, \quad b_4 = -b_5 = i = \sqrt{-1}, \end{array}\right\} \tag{22}$$

to the constants in the equations (13) or (15) and remarking that this system of constants (22) satisfies the equations of condition (14), we have the celebrated correspondence of Lie determined by the *æquationes directrices*

$$Zx + z + X + iY = 0, \quad (X - iY)x + y - Z = 0, \tag{23}$$

and studied in his well-known memoir,* "Ueber Complexe, insbesondere Linien- und Kugel-Complexe."

6. Returning now to the second case in which the quadric (9) can become a sphere for all values of k, l, m, n, namely, when all six determinants of the matrix (10) reduce to the form (12), we find that, although its equations of condition are apparently more numerous, yet it yields a more remarkable and extensive family of line-sphere contact-transformations.

In fact, the equations

$$\left.\begin{array}{l} a_i a_\sigma - a_\rho a_j = b_i b_\sigma - b_\rho b_j = c_i c_\sigma - c_\rho c_j, \\ a_i b_\sigma + b_i a_\sigma - a_\rho b_j - b_\rho a_j = 0, \\ b_i c_\sigma + c_i b_\sigma - b_\rho c_j - c_\rho b_j = 0, \\ c_i a_\sigma + a_i c_\sigma - c_\rho a_j - a_\rho c_j = 0, \end{array}\right\} \tag{24}$$

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* Mathematische Annalen, vol. 5.

which are necessary and sufficient that the determinant

$$\psi_i \psi_\sigma - \psi_\rho \psi_j \tag{25}$$

shall be changed to the form (12), possess the symmetrical solution

$$\left.\begin{array}{llll} b_i = a_i \sqrt{-1}, & b_\rho = -a_\rho \sqrt{-1}, & b_j = a_j \sqrt{-1}, & b_\sigma = -a_\sigma \sqrt{-1}, \\ c_i = a_\rho, & c_j = a_\sigma, & c_\rho = -a_i, & c_\sigma = -a_j; \end{array}\right\} \tag{26}$$

hence, the functions ψ have the forms

$$\begin{array}{l} \psi_j \quad \equiv a_j X + i a_j Y + c_j Z, \\ \psi_{j+4} \equiv c_j X - i c_j Y - a_j Z, \end{array} \quad j = 1, 2, 3, 4, \; i = \sqrt{-1}, \tag{27}$$

and, therefore, the two bilinear equations

$$x\omega_1 + y\omega_2 + z\omega_3 + \omega_4 = 0, \quad x\omega_5 + y\omega_6 + z\omega_7 + \omega_8 = 0, \tag{28}$$

where

$$\omega_i = \psi_i + d_i, \quad i = 1, 2, \ldots\ldots, 8 \tag{29}$$

determine a family of ∞^{15} transformations which change straight lines into spheres.

That these ∞^{15} transformations are contact-transformations can be established in the following manner: Combining Lie's transformation (23) expressed by the two bilinear equations

$$Zx_1 + z_1 + X + iY = 0, \quad (X - iY)x_1 + y_1 - Z = 0, \tag{30}$$

with all the transformations of the general projective group

$$\left.\begin{array}{l} \rho x_1 = \alpha_1 x + \beta_1 y + \gamma_1 z + \delta_1, \\ \rho y_1 = \alpha_2 x + \beta_2 y + \gamma_2 z + \delta_2, \\ \rho z_1 = \alpha_3 x + \beta_3 y + \gamma_3 z + \delta_3, \\ \rho \quad = \alpha_4 x + \beta_4 y + \gamma_4 z + \delta_4, \end{array}\right\} \tag{31}$$

the resulting ∞^{15} transformations form a family of line-sphere contact-transformations determined by two bilinear equations

$$\left.\begin{array}{l} (\alpha_4 X + i\alpha_4 Y + \alpha_1 Z + \alpha_3)x + (\beta_4 X + i\beta_4 Y + \beta_1 Z + \beta_4)y + \ldots\ldots = 0, \\ (\alpha_1 X - i\alpha_1 Y + \alpha_4 Z + \alpha_2)x + (\beta_1 X - i\beta_1 Y - \beta_4 Z + \beta_2)y + \ldots\ldots = 0, \end{array}\right\} \tag{31}$$

which equations are precisely of the form (28)

Hence, the equations (28) define a fifteen-parameter family of line-sphere contact-transformations (Σ) of such a nature that they are derived from the transformation (23) (Λ) of Lie and the transformations (Π) of the general projective group (30) by means of the symbolic equation

$$\Sigma = \Pi\Lambda. \tag{32}$$

7. This note limits itself to the determination of those line-sphere transformations which are defined by two bilinear equations between the point coordinates of the corresponding spaces, both spaces being ordinary; but it may be permitted to add several postscripts calling attention to other generalizations of Lie's transformation.

1°. A six-parameter family of line-sphere contact-transformations is obtained by generalizing the form*

$$\left.\begin{aligned} a &= x + yi, & b &= z + R, \\ q &= x - yi, & p &= R - z, \end{aligned}\right\} \tag{33}$$

given to Lie's transformation by Professor Darboux.

The equations

$$a_i = \varkappa_i \alpha + \lambda_i \beta + \mu_i \gamma + \nu_i \rho, \tag{34}$$

establish a correspondence between the straight lines (a_1, a_2, a_3, a_4) and the spheres $(\alpha, \beta, \gamma, \rho)$ in such a manner that intersecting straight lines are transformed into tangent spheres if the ten equations

$$\left.\begin{aligned} \varkappa_1 \varkappa_2 - \varkappa_3 \varkappa_4 = \dots = \nu_1 \nu_2 - \nu_3 \nu_4 = 1, \\ \varkappa_1 \lambda_2 + \lambda_1 \varkappa_2 - \varkappa_3 \lambda_4 - \lambda_3 \varkappa_4 = \dots = 0, \end{aligned}\right\} \tag{35}$$

are satisfied, since these equations express the necessary and sufficient conditions that the quadratic form

$$(a_1 - a_1')(a_4 - a_4') - (a_2 - a_2')(a_3 - a_3') \tag{36}$$

shall be changed by the transformation (34) into the quadratic form

$$(\alpha - \alpha')^2 + (\beta - \beta')^2 + (\gamma - \gamma')^2 - (\rho - \rho')^2. \tag{37}$$

2°. The method of the note may be employed to construct certain anomalous correspondences of a similar form in spaces of higher dimensions; these cease to be of any but formal interest in other than ordinary spaces, since for a space the number of whose dimensions is greater than three, the number of right lines is infinitely greater than the number of spheres; an example of such a correspondence in a four-dimensional space is the transformation defined by the two following bilinear equations:

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* Darboux, "Théorie des Surfaces," vol. 1, §157.

$$\left.\begin{aligned}(Z - iT)\,x + z + (X + iY)(t + 1) = 0,\\ (X - iY)\,x + y - (Z + iT)(t + 1) = 0;\end{aligned}\right\} \qquad (38)$$

it changes straight lines into spheres; it is an incomplete contact-transformation which degenerates into Lie's transformation (23) on making the fourth dimension zero.

3°. By employing the results of the note, an infinite number of infinite families of line-sphere contact-transformations of ordinary space can be constructed; these are no longer defined by two bilinear *æquationes directrices.*

PRINCETON, NEW JERSEY, *November* 1, 1899.